Optimum Isothermal Acceleration of a Plasma with Constant Magnetic Field

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On the basis of one-dimensional gasdynamics and the methods of the calculus of variations, a numerical solution has been obtained which for given inlet and outlet conditions and constant magnetic field determines the minimum length isothermal crossed-field accelerator. Since the results are presented in a nondimensional form, it is possible to design many MHD accelerators with a variety of operating conditions. By comparing the optimum solution with various other isothermal solutions, it can be shown that considerable reductions in length are possible. The analysis thus indicates that there is much to be gained from improved accelerator design.

Nomenclature

ρ = mass density
u = flow velocity
A = cross-sectional area of channel
p = plasma pressure
j = current density
B = magnetic induction
σ = conductivity
E = electric field strength
R = gas constant
T = gas temperature
x = axial distance along channel
$\dot{m}$ = mass flow rate
ρ^* = ρ/ρ_0
r = as defined in Eq. (12)
u^* = $u/(RT)^{1/2} = \gamma^{1/2} M$
x^* = $\sigma B^2 x/\rho_0(RT)^{1/2} = \sigma B^2(RT)^{1/2} x/p_0$
j^* = $j/\sigma B(RT)^{1/2}$
p^* = p/p_0
A^* = A/A_0

Introduction

In recent years there has been considerable interest in the development of a steady-state d.c. crossed field plasma accelerator for propulsion applications. This interest stems primarily from the expectancy that such devices can produce specific impulses that are attractive for many space applications. Unfortunately, very little has been done, either theoretically or experimentally, to optimize the design of such devices. The study in the present paper is concerned with the optimum design of a crossed-field accelerator which allows no variations in plasma temperature with length. Although this requirement is very stringent, there is much that can be said in favor of such an approach. Isothermal acceleration insures that added electrical energy appears as kinetic and not as thermal energy and allows minimum losses due to heat transfer, dissociation, and ionization consistent with adequate gas conductivity. In addition, large temperature variations do not seem practical because of the temperature limitations of wall materials. Once the temperature is high enough for adequate electrical conductivity, any further increases will result in reduced efficiency from thermal leaving losses and increased wall erosion.

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Assumptions

In order to obtain a theoretical solution to the equations of magnetogasdynamics, the usual one-dimensional flow approximations are made. The magnetic field is assumed to be a constant and perpendicular to an applied electric field that is varied along the length of the channel. It is assumed that the required electric fields may be obtained either by segmenting the electrodes and/or varying the spacing between electrodes. The cross-sectional area of the channel is allowed to vary with axial distance. This can be accomplished by varying the width of the channel. The variations in electric field and area distribution are not independent, since they must satisfy the condition of constant temperature. Thus, a delicate balance between Joule heating and thermal expansion is necessary to maintain isothermal acceleration. In addition, the plasma is assumed to be a thermally perfect gas. The electrical conductivity is assumed constant and scalor, which implies that the variations with pressure and current are small and Hall effects are negligible or can be eliminated by segmenting the electrodes. The effects of wall losses and ion slip are neglected. These assumptions impose certain restrictions on the magnetic field strength, size, and power level.

The significance of the forementioned statements should be realized, but a full discussion of the assumptions and the derivation of the equations is not possible in a paper of this scope. The assumptions are not new, however, and the reader is referred to earlier works on plasma acceleration.[1, 2]

Analysis

The one-dimensional equations of motion governing the steady flow of a nonviscous, nonheat conducting, perfect gas plasma at constant temperature, together with Ohm's law for scalor conductivity, are as follows†:

State

$$p = \rho RT \tag{1}$$

Continuity

$$\rho u A = \dot{m} = \text{const} \tag{2}$$

Momentum

$$\rho u du + dp = jBdx \tag{3}$$

Energy

$$udp = -j^2/\sigma dx \tag{4}$$

† Meter-kilogram-second units are used throughout.

Ohm's Law

$$j = \sigma(E - uB) \tag{5}$$

Since the forementioned equations contain more unknowns than equations, one is free to specify several quantities or to impose suitable side conditions or equations that will determine a solution. Thus, there are infinitely many isothermal solutions, several of which have already been obtained.[2–4] The condition of constant temperature, which already has been imposed, however, can prove very restrictive and many of the possible solutions result in very long accelerator channels or in a very rapid loss of pressure. For this reason it is desirable to impose the condition that the length be a minimum for a given exit velocity and pressure or that the exit pressure be maximum for a given length and exit velocity. The formulation of the problems for minimum length and maximum pressure can be shown to be equivalent, since they lead to the same set of differential equations. The problems of minimum exit area or entropy increase and maximum thrust per unit exit area are likewise all equivalent to the solution for minimum length.

To formulate the problem of minimum length, first combine Eqs. (3–5) to obtain

$$\rho u^2 du = jEdx \tag{6}$$

Since the product jE is the power input per unit volume and ρuA is the mass flow, it is easy to show from Eq. (6) that all electrical energy is converted to gas kinetic energy for any isothermal accelerator.

If the current density is eliminated from Eq. (6) by substituting from Eq. (5), one can write

$$\rho du = \sigma B^2 \beta(\beta - 1)dx \tag{7}$$

where $\beta = E/uB$. An alternate form in terms of the electric field is

$$\rho u^2 du = \sigma E^2[(\beta - 1)/\beta]dx \tag{7a}$$

Similarly, by substituting for current and pressure from Ohm's law and the equation of state and the differential of x from Eq. (7) the energy equation becomes

$$d\rho/\rho = -[(\beta - 1)/\beta RT]udu \tag{8}$$

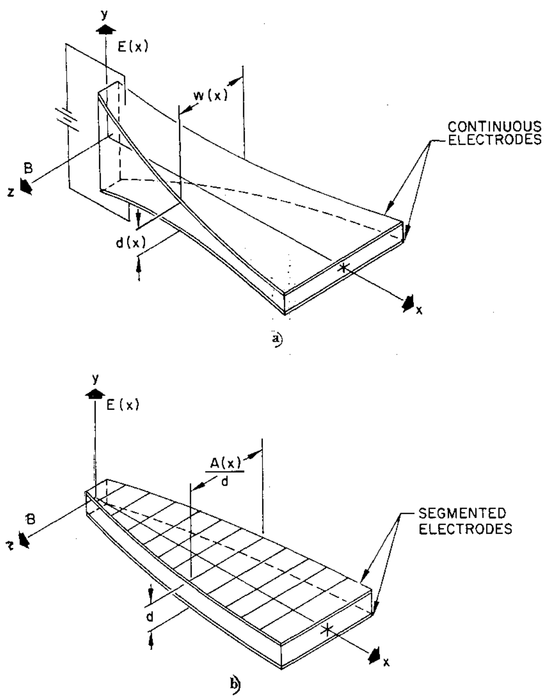

Fig.1 Possible channel and electrode configurations for a) continuous electrodes and b) segmented electrodes.

Eliminating the parameter β between Eqs. (7) and (8), solving for the differential of x, and integrating, one obtains two alternate integral forms for the accelerator length (primes denote differentiation with respect to u):

$$L = -\int_{u_0}^{u_e} \frac{1}{\sigma B^2 RT} \frac{\rho^2}{\rho'} \left(1 + \frac{RT}{u} \frac{\rho'}{\rho}\right)^2 udu \tag{9}$$

or

$$L = -\int_{u_0}^{u_e} \frac{1}{\sigma E^2 RT} \frac{\rho^2}{\rho'} u^3 du \tag{9a}$$

If either the electric or magnetic field is specified as a function of velocity, the problem of minimum length is evidently a problem in the calculus of variations in which one wishes to determine the dependence of density on velocity which will make either of the forementioned integrals a minimum. The simplest cases and, perhaps, the most convenient experimentally are constant electric field or constant magnetic field.‡ Once the optimum density distribution is determined, the optimum electric or magnetic field corresponding to this solution is found by solving for β from Eq. (8):

$$\beta = \frac{1}{1 + (RT/u)(\rho'/\rho)} \tag{10}$$

Although the assumption of constant conductivity is not essential, the solution is more simply obtained and can be suitably nondimensionalized so as to be independent of the choice of gas mixture and temperature. It is possible, however, to express the conductivity as a function of density at constant temperature[3, 5, 6] or even to include the nonequilibrium effects of current density.[7] Considerable reductions in length may be possible by taking advantage of these effects. Such refinements, however, would necessitate the choice of a particular gas, seed material, and temperature.

Assuming then that σ, B, and T are all constants, a necessary condition for a minimum of the integral given in Eq. (9) is that ρ satisfy the associated Euler differential equation (sufficiency is assumed):

$$\frac{d}{du}\left[\frac{(RT)^2}{u} - \left(\frac{\rho}{\rho'}\right)^2 u\right] + 2u\left(\frac{\rho}{\rho'}\right) + 2RT = 0 \tag{11}$$

This, together with all previous equations, determines a solution. It is advantageous to rewrite the necessary equations in dimensionless form by introducing suitable dimensionless variables.§ In addition, it was found (for the numerical methods chosen) that it was desirable to reduce Eq. (11) (which is of second order) to two simultaneous first-order equations. All equations are then of the first order or algebraic. The resulting equations are

$$r \equiv (\rho^*)'/\rho^* \tag{12}$$

$$r' = \frac{r}{2u^*} + r^2 + \left[\frac{1}{2(u^*)^3} + \frac{1}{u^*}\right] r^3 \tag{13}$$

$$x^{*\prime} = -\rho^*\left(\frac{u^*}{r} + \frac{r}{u^*} + 2\right) \tag{14}$$

$$\beta = \left(1 + \frac{r}{u^*}\right)^{-1} \tag{15}$$

$$A^* = \frac{1}{\rho^*}\left(\frac{u_0^*}{u^*}\right) \text{ (subscripts zero refer to inlet conditions)} \tag{16}$$

‡ It is not possible to optimize either the electric or magnetic fields as a function of velocity, since this leads to infinite fields.

§ Quantities with asterisk are the dimensionless counterparts of previously defined symbols. Primes denote differentiation with respect to u^*.

$$j^* = u^*(\beta - 1) \tag{17}$$

$$p^* = \rho^* \tag{18}$$

Given the initial conditions β_0, u_0^*, ρ_0^*, and x_0^*, Eqs. (12–18) determine the solution. These equations were numerically integrated on an IBM 7090 digital computer. Inlet conditions were taken as

$$\begin{aligned} \beta_0 &= \text{(several values)} \\ u_0^* &= (\tfrac{5}{3})^{1/2}(M_0 = 1 \text{ for } \gamma = \tfrac{5}{3}) \\ \rho_0^* &= 1 \\ x_0^* &= 0 \end{aligned}$$

If the electric field is assumed to be obtained by segmented electrodes placed along flat parallel planes, then A^* can be interpreted as the shape of the side walls. If, on the other hand, the electric field is obtained by shaping the electrodes at a constant potential, the approximate shape can be calculated by assuming that $V = Ed$, where V is the voltage and d the spacing between electrodes. The width must then vary to provide the same area distribution (see Fig. 1). In the nondimensional form one can write

$$d^* = \frac{d}{d_0} = \frac{\beta_0}{\beta}\left(\frac{u_0^*}{u^*}\right) = \frac{E_0}{E} = \frac{1}{E^*} \tag{19}$$

$$w^* = \frac{w}{w_0} = \frac{1}{\rho^*}\left(\frac{\beta}{\beta_0}\right) \tag{20}$$

The results of the computer calculations are shown in graphical form in Figs. 2–6, showing the variation of pressure or density ratio ρ^* or p^*, dimensionless velocity u^*, area ratio A^*, dimensionless current density j^*, electrode spacing ratio d^* or reciprical of electric field ratio $1/E^*$, and width ratio w^* with dimensionless length x^*.

Power and Thrust

The total power required for the accelerator plus generator is given by the product of the mass flow and the total enthalpy. In terms of dimensionless variables, this is

$$P = p_0 A_0 u_0^* (RT)^{1/2}\left[\frac{\gamma}{\gamma - 1} + \frac{u_e^{*2}}{2}\right] \tag{21}$$

The thrust is the product of mass flow and exhaust velocity and can be written as

$$F = p_0 A_0 u_0^* u_e^* \tag{22}$$

Discussion

The solutions obtained determine isothermal accelerator designs that are optimum in the sense that they are the shortest, or that they have the least pressure drop or smallest exit area for a given length. In theory a device may be made as short as desired by choosing β_0 as large as necessary. In practice this is not possible, since if β_0 is too large, the current at the inlet will exceed the limits of the cathode material. In addition, the outlet pressure may be sufficiently low that Hall effects cannot be controlled or the exit area may be prohibitively large.

In order to illustrate some of these effects, suppose an accelerator producing a specific impulse of 2500 sec is required. This corresponds to an exhaust velocity of 24,500 m/sec, which, in turn, determines u_e^* for any given plasma temperature and β_0. Using Fig. 3, one can find the length x_e^*. The resulting accelerator lengths as a function of temperature and β_0 are given in Fig. 7 where it has been assumed that $B = 1$ weber/m^2 and that the plasma is helium seeded with 1% cesium. The inlet pressure is taken as 10^5 newtons/m^2 (about 1 atm); the inlet Mach number is unity. The assumed average conductivities are given in Fig. 8; the corresponding pressure ratios are shown in Fig. 9.

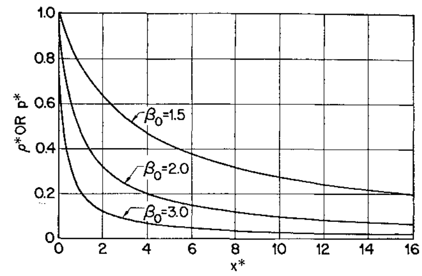

Fig. 2 Pressure or density vs axial distance.

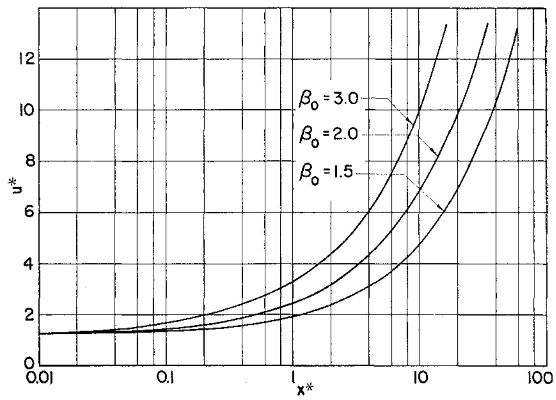

Fig. 3 Velocity vs axial distance.

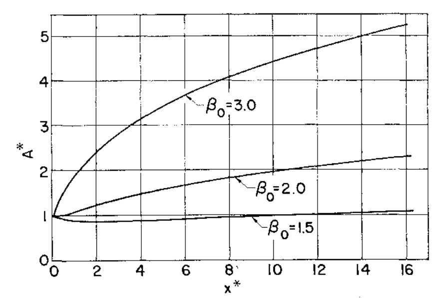

Fig. 4 Channel cross-sectional area vs axial distance.

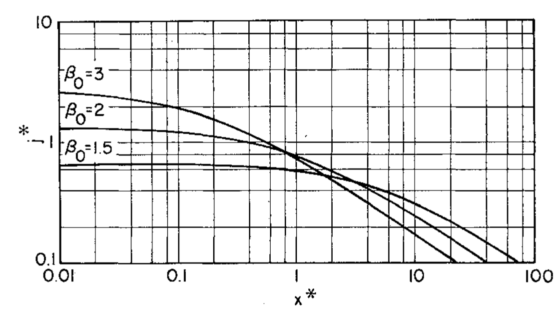

Fig. 5 Current density vs axial distance.

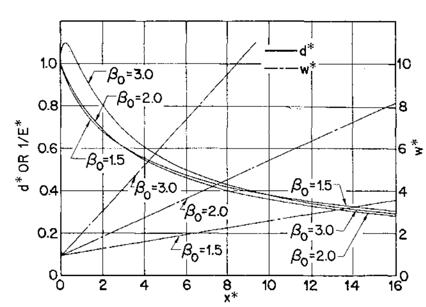

Fig. 6 Channel height and width for continuous electrodes or the electric field distribution for segmented electrodes ($d^* = 1/E^*$).

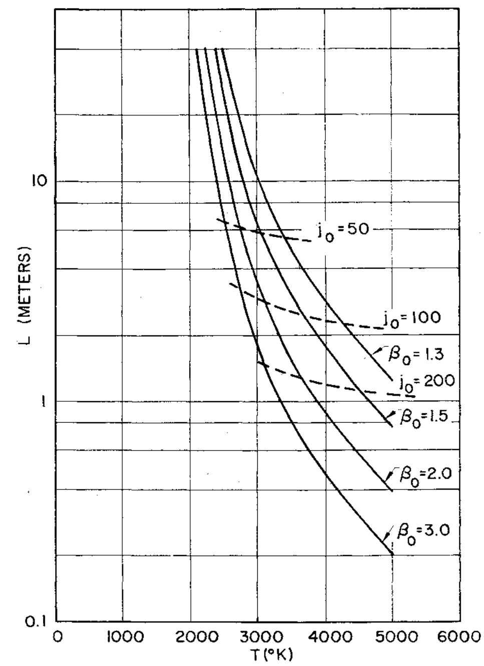

Fig. 7 Optimum accelerator lengths for I_{sp} = 2500 sec, helium seeded with 1% cesium, p_0 = 10^5 newtons/m^2, M_0 = 1, and B = 1 weber/m^2 with lines of constant maximum current density, j_0, in amp/cm^2.

From this example it is clear that shorter lengths tend to require higher current, lower inlet pressure, or higher magnetic field strength. It is interesting to note that increasing the temperature does little to decrease the length at a constant maximum current, but greatly reduces the pressure drop across the accelerator.

Whether or not satisfactory lengths can be obtained in a practical system is not yet clear and certainly cannot be decided on the basis of these preliminary results alone. It can be seen, however, that certain compromises limit the choice of design parameters. Just how short a particular accelerator must be depends largely upon the application. In general, the MHD accelerator is best suited for very large devices. For example, Ref. 8 proposes a device capable of launch from the earth's surface. For this application, a length of 10 m is not excessive. On the basis of the results of this paper, it

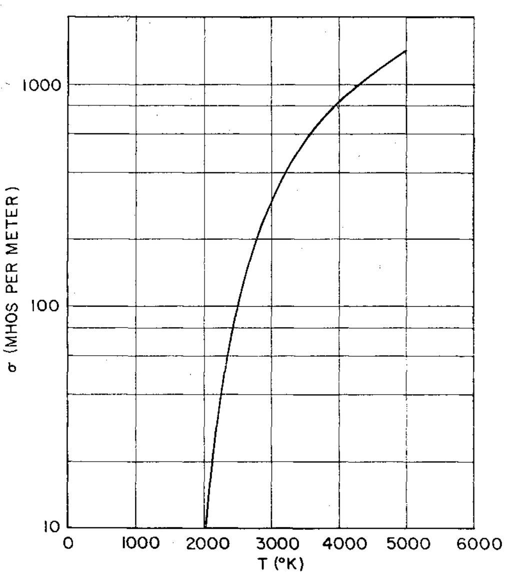

Fig. 8 Assumed average conductivity for helium seeded with 1% cesium used to compute lengths in Fig. 7 and pressures in Fig. 9.

does not appear at all difficult to design a device that will meet this length requirement. On the other hand, for low thrust applications, the cross section of the channel must be quite small and the length correspondingly shorter in order to reduce losses to the walls of the channel. Thus, low thrust devices will require higher currents, higher temperatures, and longer life. These problems can be reduced if it proves possible to decrease the pressure or increase the magnetic field without producing serious Hall effects. The lengths in Fig. 7 could be reduced an order of magnitude if the pressure were reduced to 0.1 atm at the inlet. This could be done without changing the current requirements of the cathode.

As a final consideration, a comparison of the optimum solution with other possible solutions is in order. Since there are infinitely many possible isothermal accelerators, the comparison can not be exhaustive, and since the solution obtained is optimum only for the case of constant magnetic field, as examples, three solutions that satisfy this requirement will be considered. These are constant area,[3] constant β,[4] and constant current ($j^* = 1$).

The case of constant area is well known and of considerable interest experimentally. The solution, in terms of the same dimensionless variables used in the present analysis, is

$$\frac{x^*}{\gamma^{1/2}} = \frac{(u^*)^2}{2} - 2\ln\frac{u^*}{\gamma^{1/2}} - \frac{1}{2}(u^*)^2 - \frac{(\gamma^2 - 1)}{2\gamma} \qquad (23)$$

$$\rho^* = \rho^* = \frac{u_0^*}{u^*} = \frac{\gamma^{1/2}}{u^*} \qquad (\text{for } M_0 = 1) \quad (24)$$

The solution for constant β given by Sutton[4] can be written

$$\frac{(\beta - 1)\beta x^*}{\gamma^{1/2}} = \frac{\exp\{[(\beta - 1)/2\beta](u_0^*)^2\}}{u_0^*} \times \int_{u_0^*}^{u^*} \exp\left[\frac{-(\beta - 1)}{2\beta}(u^*)^2\right] du^* \quad (25)$$

$$\rho^* = p^* = \exp\{-(\beta - 1)/2\beta[(u^*)^2 - (u_0^*)^2]\} \quad (26)$$

The solution for $j^* = 1$ is obtained easily by performing the integrations of Eqs. (9) and (15) with the aid of Eq. (17) and with j^* set equal to unity. The results are

$$x^* = \frac{e^{u_0^*}}{1 + u_0^*}\{-e^{-u^*}[(u^*)^2 + 2u^* + 2] + e^{-u_0^*}[(u_0^*)^2 + 2u_0^* + 2]\} \quad (27)$$

$$\rho^* = p^* = \frac{(1 + u^*)}{(1 - u_0^*)} \exp(u_0^* - u^*) \quad (28)$$

From each of these cases an example was chosen with u_e^* = 6 and compared with the optimum solution with u_e^* = 6 and at the same exit pressure ratio. The results of the comparison are as follows:

constant area		optimum
$p^* = 0.215$	$\leftrightarrow$	$p^* = 0.215$
$x^* = 18.6$		$x^* = 17.2$
constant β		optimum
$p^* = 0.1$	$\leftrightarrow$	$p^* = 0.1$
$x^* = 40.2$		$x^* = 8.0$
$j^* = 1$		optimum
$p^* = 0.028$	$\leftrightarrow$	$p^* = 0.028$
$x^* = 2.5$		$x^* = 0.4$

In all of the forementioned cases the optimum accelerator has the shortest length, as would be expected. The examples for the constant β and j^* are clearly far from optimum since the optimum is only about one-fifth or one-sixth as long as these, respectively. For the case of constant area, however, only slight reductions in length are possible at the same pressure ratio. The reason for this is that the optimum solution is, in fact, very nearly constant area at the same pressure ratio.

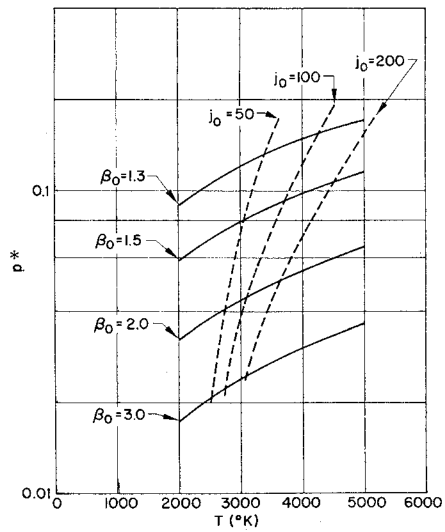

Fig. 9 Optimum pressure ratios for I_{sp} = 2500 sec, helium seeded with 1% cesium, $p_0 = 10^5$ newtons/m², M_0 = 1, and B = 1 weber/m² with lines of constant maximum current density, j_0, in amp/cm².

It corresponds to $\beta_0 \simeq 1.5$, for which A^* (Fig. 4) varies only slightly from unity. If a lower exit pressure is acceptable, however, β_0 can be chosen larger and an expanding channel can be designed which is much shorter. This would require higher currents, but not as high as the constant area case, since it is found that the maximum current required by the constant area case is about three times higher than the optimum.

Conclusions

A numerical solution has been obtained, which, for given inlet and outlet conditions and constant magnetic field determines the minimum length isothermal crossed field accelerator. Although the solution does not exhaust all of the possibilities for isothermal acceleration, it demonstrates that, for the interesting case of constant magnetic field, considerable reductions in length are possible. The results easily can be extended to other field shapes and can be refined to include the variation of electrical conductivity with pressure and current density.

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Nonequilibrium-Dissociated Boundary Layers with a Reacting Inviscid Flow

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This paper examines the problem of nonequilibrium boundary layers when chemical reaction occurs along the inviscid flow streamlines at the outer edge for the case of high Reynolds number. A generalized species conservation equation in similarity coordinates, which explicitly shows the influence of nonequilibrium inviscid flow reaction on the boundary layer, is derived and discussed. Except in the special case of a highly cooled body immersed in a slow, hot, dissociatively relaxing gas stream, it is shown that the inviscid reaction rate can be neglected in analyzing nonequilibrium boundary-layer flows encountered in practice, unless the details of the gas-state profiles near the outer edge of the boundary layer are of interest.

Nomenclature

α = atom mass fraction
$\bar{c}_p$ = frozen specific heat of mixture
η = similarity coordinate, Eq. (4)
f = stream function in similarity coordinates
g = net reaction rate function, Eq. (2)
Γ = characteristic (flow time/reaction time) parameter, Eq. (8)
G,H = nondimensional reaction rate distribution functions, Eqs. (5–7)
h_D = specific dissociation energy of molecules
κ = $x(d \ln \xi / dx)$
k_R' = recombination rate constant
μ = coefficient of viscosity
ω = recombination rate temperature-dependence exponent
p = static pressure
ρ = mixture density
r_B = local body radius
$\mathcal{R}$ = universal gas constant
ξ = similarity coordinate, Eq. (4)
θ = T/T_e

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